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Binary Conditions for Double and Triple Points on a Cubic.[*]

BY LEROY A. HOWLAND.

The cubic plane curve has been exhaustively studied by use of the theory of ternary forms. Conditions necessary and sufficient for the existence of one or more double points, of a triple point, etc., have been derived in terms of ternary invariants, chiefly by Aronhold, Gordan, Clebsch and Gundelfinger.[†]

The theory of binary forms, however, may also be applied to the study of the cubic. Compare, for example, the treatment of the inflexions of a cubic by Clebsch[‡] and by the writer.[§] Conditions that a cubic degenerate into a conic and a straight line or into three straight lines (non-current), *i. e.*, conditions for two or three double points, were derived by Brioschi, expressed in terms of the simultaneous invariants of two binary forms.[||] In this paper conditions of the same nature will be derived for the case of one double point (node or cusp) and for the case of a triple point.

The argument will be based upon two properties of the polars of algebraic plane curves. Let P be a point of the curve C_n, and let Q be a point of the plane not lying on a tangent to C_n at P.

PROPERTY I. *A necessary and sufficient condition that P be an r-fold point for C_n is that the first $r—1$ polars of C_n with respect to Q pass through P.*

That it is necessary follows at once from the well-known

THEOREM A. *An r-fold point of C_n is an $(r—s)$-fold point of the s-th polar $(s<r)$ of any other point.*

To prove it sufficient we will take the vertices III and I of the frame of reference to be the points P and Q respectively. The equation of the curve will then be

$$x_3^{n-1} u_1 + x_3^{n-2} u_2 + \dots + x_3 u_{n-1} + u_n = 0,$$

and the equations of the polars will be

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[*] Read before the American Mathematical Society, April 27, 1912.

[†] Cf. Clebsch-Lindemann, *Vorlesungen über Geometrie,* Bd. I, p. 580 ff.

[‡] "Theorie der Binären Formen," pp. 234–254.

[§] Dissertation, München, 1908.

[||] "Sulle condizioni per la decomposizione di una forma cubica ternaria in tre fattori lineari," *Annali di Matematica,* (2) VII (1875), pp. 189–192. "Sulle condizioni per la decomposizione di una cubica in una conica ed in una retta," *Acc. R. d. L.,* (2) III (1876), pp. 89–90.

$$x_3^{n-1}\frac{\partial u_1}{\partial x_1}+x_3^{n-2}\frac{\partial u_2}{\partial x_1}+\ldots\ldots+\frac{\partial u_n}{\partial x_1}=0,$$

$$x_3^{n-2}\frac{\partial^2 u_2}{\partial x_1^2}+x_3^{n-3}\frac{\partial^2 u_3}{\partial x_1^2}+\ldots\ldots+\frac{\partial^2 u_n}{\partial x_1^2}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_3^{n-r+1}\frac{\partial^{r-1} u_{r-1}}{\partial x_1^{r-1}}+\ldots\ldots+\frac{\partial^{r-1} u_n}{\partial x_1^{r-1}}=0.$$

Since by hypothesis all these polars pass through III, we have

$$\frac{\partial u_1}{\partial x_1}=\frac{\partial^2 u_2}{\partial x_1^2}=\frac{\partial^3 u_3}{\partial x_1^3}=\ldots\ldots=\frac{\partial^{r-1} u_{r-1}}{\partial x_1^{r-1}}=0.$$

This means that u_k is divisible by x_2 for $k=1,2,\ldots\ldots,r-1$. But the first non-vanishing u may not be divisible by x_2, otherwise I would be on a tangent to C_n at III, contrary to hypothesis. Consequently,

$$u_1\equiv u_2\equiv u_3\equiv\ldots\ldots\equiv u_{r-1}\equiv 0,$$

and C_n has an r-fold point.

PROPERTY II. *A necessary and sufficient condition that s of the r branches of C_n through P be tangent at P to a line t, is that C_n and its first s—1 polars with respect to Q be tangent to t at P.*

To prove the condition sufficient we choose the frame of reference as before, taking, in addition, the side I to be the line t. The equations of the curve and its polars are then

$$x_3^{n-r}u_r+x_3^{n-r-1}u_{r+1}+\ldots\ldots+u_n=0,$$

$$x_3^{n-r}\frac{\partial u_r}{\partial x_1}+\ldots\ldots+\frac{\partial u_n}{\partial x_1}=0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$x_3^{n-r}\frac{\partial^{s-1} u_r}{\partial x_1^{s-1}}+\ldots\ldots+\frac{\partial^{s-1} u_n}{\partial x_1^{s-1}}=0.$$

Assume

$$u_r=a_0x_1^r+r\,a_1x_1^{r-1}x_2+\ldots\ldots+r\,a_{r-1}x_1x_2^{r-1}+a_rx_2^r.$$

Then

$$\frac{\partial^k u_r}{\partial x_1^k}=r\,(r-1)\ldots\ldots(r-k+1)\,[a_0x_1^{r-k}+(r-k)\,a_1x_1^{r-k-1}x_2+\ldots\ldots+a_{r-k}x_2^{r-k}].$$

By hypothesis this must be divisible by x_1 for $k=0,1,\ldots\ldots,s-1$. That is, $a_{r-k}=0$ for $k=0,1,2,\ldots\ldots,s-1$.

Hence,

$$u_r = a_0 x_1^r + r\, a_1 x_1^{r-1} x_2 + \ldots . + \frac{r(r-1)\ldots .(s+1)}{(r-s)\,!}\, a_{r-s} x_1^s x_2^{r-s},$$

which proves the theorem.

That the condition is necessary is well known, and is indeed obvious, if we write down the equation of C_n referred to the frame of reference chosen above.

Consider the cubic

$$y_3^3 + U_1 y_3^2 + U_2 y_3 + U_3 = 0.$$

The transformation with determinant equal to 1,

$$x_1 = y_1, \qquad x_2 = y_2, \qquad x_3 = \tfrac{1}{3} U_1 + y_3,$$

carries this over into

$$x_3^3 - 3 f x_3 + 2 \phi = 0, \tag{1}$$

where f and ϕ are forms of the second and third degree respectively in x_1 and x_2. If the line

$$x_3 = \xi_x \tag{2}$$

cuts the curve at P in three coincident points, we have

$$\xi_x^3 - 3 f \xi_x + 2 \phi \equiv \eta_x^3. \tag{3}$$

A necessary and sufficient condition that P be a double point is (Property I) that the polar of III pass through P; *i. e.*, that

$$\xi_x^2 - f \text{ be divisible by } \eta_x, \tag{4}$$

or, representing f and ϕ symbolically by a_x^2 and α_x^3, that

$$(\xi \eta)^2 - (a \eta)^2 = 0. \tag{5}$$

We shall use the following simultaneous invariants of f and ϕ, the notation being that of Clebsch:

$D = (a b)^2$, discriminant of f;

$\Delta = (\alpha\beta)^2 \alpha_x \beta_x$, Hessian of ϕ;

$R = (\Delta \Delta')^2$, discriminant of Δ and of ϕ;

$P = (a \alpha)^2 \alpha_x$;

$E = (a \Delta)^2$;

$F = (a p)^2$;

$F - 2 D E$ is the resultant of f and ϕ.

We shall also use the following abbreviations:

$$K = (a\xi)^2, \quad L = (a\xi)(a\eta), \quad M = (a\eta)^2, \quad N = (\xi\eta)^2.$$

Using a well-known symbolic identity, we find

$$Nf = K\eta^2 - 2L\xi\eta - M\xi^2. \tag{6}$$

Combining (3) and (6), we have

$$2N\phi = N\eta^3 + 3K\xi\eta^2 - 6L\xi^2\eta + (3M - N)\xi^3. \tag{7}$$

Building the invariants of f and ϕ by means of (6) and (7), we find

$$\left.\begin{aligned} DN &= 2(KM - L^2), \\ 2p &= (2D - K)\xi + M\eta, \\ 4F &= K^3 + M^3 - 2KLM + 4D(ML - K^2) + 4D^2K, \\ 2E &= KD + LM - K^2 - LN, \\ 8R &= 4K^3 + 8L^3 - 12KLM - 6DK^2 + 12KLN - 12DLN - 9M^2N \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 6MN^2 - N^3. \end{aligned}\right\} \tag{8}$$

The quantities

$$\left.\begin{aligned} A &= 8E - D^2, \\ B &= 20DE - 8R - 16F + D^3 \end{aligned}\right\} \tag{9}$$

have the values

$$A = 4L(M - N) - (2K - D)^2, \tag{10}$$

$$B = 6L(M - N)(2K - D) - (4M - N)(M - N)^2 - (2K - D)^3,^* \tag{11}$$

$$\begin{aligned} A^3 + B^2 = (M - N)^2 [&64L^3(M - N) - 12L^2(2K - D)^2 + (4M - N)^2(M - N)^2 \\ &- 12L(4M - N)(M - N)(2K - D) + 2(4M - N)(2K - D)^3]. \end{aligned}$$

By use of the first equation (8), we can change this into

$$\begin{aligned} A^3 + B^2 = (M - N)^3 [&16L(2D - K)(M - N) - 4L(2K - D)(4M - N) \\ &+ 16L^3 + (4M - N)^2(M - N) - 4(2K - D)^2(2D - K)]. \end{aligned} \tag{12}$$

A necessary condition for a double point is then

$$A^3 + B^2 = 0. \tag{13}$$

It does not appear to be sufficient, for when $A^3 + B^2$ is zero, it may happen that $M - N$ is not zero, but rather the quantity in brackets. (13) is not a sufficient condition that P be a double point; it is, however, sufficient that the curve have a double point, as we can show by using some results of Clebsch.

Clebsch shows that if $x_3 = \xi_x$ is a line which cuts the curve in three coincident points, the other lines having this property are

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* Equations (6)-(11) are taken, with some change of form and corrections for misprints, from Clebsch, *loc. cit.*, p. 241.

$$x_3 = \xi_x + z,$$
$$z = \frac{-3(\xi - m^2\eta) \pm \sqrt{9(\xi - m^2\eta)^2 - 12(1-m^3)(\xi^2 - m\eta^2 - f)}}{2(1-m^3)}, \qquad (14)$$

where m is a root of the equation

$$(M-N)m^4 - 2(2K-D)m^3 + 6Lm^2 - (4M-N)m + (K-2D) = 0.^{*} \qquad (15)$$

The two invariants of this biquadratic are

$$i = 3DN - 6(KM - L^2),$$
$$j = -\tfrac{3}{8}[16L(2D-K)(M-N) - 4L(2K-D)(4M-N) + 16L^3 + (M-N)(4M-N)^2 - 4(2K-D)^2(2D-K)].$$

$i = 0$ by reason of the first of equations (8). Hence, if j is zero, the equation (15) has a triple root, (14) gives only four values of z and there are but five lines which meet the curve in three coincident points. This can happen only when the curve has a double point. Therefore, (13) is a sufficient condition.

We have, however, overlooked the possibility that the curve may have a double point the tangent at which goes through III. Any line through III has the form

$$\zeta_x \equiv \zeta_1 x_1 + \zeta_2 x_2 = 0.$$

Both ζ_1 and ζ_2 can not be zero. Let us suppose $\xi_1 \neq 0$. If this line cuts the curve in three coincident points, then

$$\zeta_1^3 x_3^3 - 3\zeta_1 (a\zeta)^2 x_2^2 x_3 + 2(\alpha\zeta)^3 x_2^3$$

must be a cube and hence its Hessian must vanish identically; *i. e.*,

$$\zeta_1^4 (a\zeta)^2 = 0, \qquad 2\zeta_1^3 (\alpha\zeta)^3 = 0,$$

or f and ϕ have a common factor, and $\zeta_x = 0$ cuts the curve where $x_3 = 0$. For a double point it is necessary and sufficient that the three polars

$$x_3^2 - f = 0,$$
$$-3x_3 \frac{\partial f}{\partial x_1} + 2\frac{\partial \phi}{\partial x_1} = 0,$$
$$-3x_3 \frac{\partial f}{\partial x_2} + 2\frac{\partial \phi}{\partial x_2} = 0$$

pass through it. Hence, ζ_x must also be a factor of $\frac{\partial \phi}{\partial x_1}$ and $\frac{\partial \phi}{\partial x_2}$, and therefore a double factor of ϕ. The conditions that f and ϕ have a common factor, which is a double factor of ϕ, are easily seen to be

$$R = E = F = 0.$$

In this case $A = -D^2$, $B = D^3$ and $A^3 + B^2 = 0$.

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[*] *Loc. cit.*, p. 240; or for the form used here, cf. Howland, Dissertation, p. 11 ff.

THEOREM. *The necessary and sufficient condition that the cubic* (1) *have a double point, is* $A^3+B^2=0$.

For a cusp, the tangent at which does not go through III, we must have (Property II) $x_3=\xi_x$ tangent to $x_3^2-f=0$, *i. e.*, ξ_x^2-f must be divisible by η_x^2. Hence, in addition to $M-N=0$, the discriminant of ξ_x^2-f must vanish, or $2(a\xi)^2-(ab)^2=2K-D=0$.

In this case we see from equations (10) and (11) that $A=B=0$. Conversely, if $A=B=0$, we have a double point and hence $M-N=0$. (10) shows that then $2K-D=0$ also and the point is a cusp.

If the tangent passes through III, we have shown that the point must be the point $\zeta_2:-\zeta_1:0$ and that $R=E=F=0$.

The equation of the tangents at this point is

$$2\phi_{11}x_1^2+4\phi_{12}x_1x_2+2\phi_{22}x_2^2-6f_1x_1x_3-6f_2x_2x_3=0,$$

$\zeta_2, -\zeta_1$ being substituted for x_1, x_2 in the coefficients.

For a cusp this must be a constant multiple of ζ_x^2, and hence

$$\phi_{11}\phi_{22}-\phi_{12}^2=0, \quad f_1=0, \quad f_2=0.$$

The first equation says that the Hessian of ϕ has the factor ζ_x. This gives nothing new, for, as is well known, a double factor of ϕ is also a double factor of its Hessian. The other two equations, however, tell us that f has a double factor ζ_x. Consequently, $D=0$, and by equation (9), $A=B=0$.

THEOREM. *The necessary and sufficient conditions that the cubic* (1) *have a cusp are* $A=B=0$.

Instead of taking up next the case of a triple point of a cubic, we shall consider a more general problem, for it is possible by this method to handle the problem of an n-fold point on a curve of the n-th degree.

The equation of the curve being

$$y_3^n+v_1y_3^{n-1}+v_2y_3^{n-2}+\ldots\ldots+v_{n-1}y_3+v_n=0,$$

we can transform it by the transformation with determinant 1,

$$x_1=y_1, \qquad x_2=y_2, \qquad x_3=\frac{1}{n}v_1+y_3,$$

into

$$x_3^n+\frac{n(n-1)}{2!}u_2x_3^{n-2}+\frac{n(n-1)(n-2)}{3!}u_3x_3^{n-3}+\ldots\ldots+n\,u_{n-1}x_3+u_n=0. \quad (16)$$

The polars of this curve with respect to the vertex III are:

$$
\begin{aligned}
&x_3^{n-1}+\frac{(n-1)(n-2)}{2!}u_2x_3^{n-3}+\ldots.+u_{n-1}=0,\\
&x_3^{n-2}+\frac{(n-2)(n-3)}{2!}u_2x_3^{n-4}+\ldots.+u_{n-2}=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&x_3^3+3u_2x_3+u_3=0,\\
&x_3^2+u_2=0,\\
&x_3=0.
\end{aligned}
$$

If this curve has an n-fold point P, then, by Theorem A, this point is $(n-1)$-fold for the first polar, $(n-2)$-fold for the second, ..., a double point for the $(n-2)$-th polar and a simple point for the $(n-1)$-th. It follows directly from this that u_2, u_3, ..., u_{n-1}, u_n are constant multiples of the second, third, ..., $(n-1)$-th, n-th powers of the same linear form, $a_x=a_1x_1+a_2x_2$.

These conditions are evidently sufficient, for the equation then has the form

$$x_3^n+\alpha a_x^2x_3^{n-2}+\beta a_x^3x_3^{n-3}+\ldots.+\nu a_x^n=0, \tag{17}$$

and any line through the point $-a_2:a_1:0$,

$$x_3+\lambda a_x=0,$$

obviously has with this curve at P a multiplicity of intersection equal to n.

The equations of the lines into which (17) degenerates are

$$x_3=\lambda_i a_x, \qquad (i=1,2,\ldots.,n),$$

where

$$\Sigma\lambda_i=0,\quad \Sigma\lambda_i\lambda_j=\alpha,\ \ldots.,\ \lambda_1\lambda_2\ldots.\lambda_n=(-1)^n\nu,$$

or, in other words, the λ's are roots of the equation

$$\lambda^n+\alpha\lambda^{n-2}+\beta\lambda^{n-3}+\ldots.+\nu=0.$$

a_x may be found as the square root of u_2. In case u_2 vanishes identically, a_x may be found as the quotient of u_i by u_{i-1}, or, if there is no pair of consecutive non-vanishing u's, as the quotient of u_i by $\frac{\partial u_i}{\partial x_1}$.

$H(u_p)\equiv 0$ is a necessary and sufficient condition that u_p be a p-th power, $H(u_p)$ being the Hessian of u_p. $H(u_p)\equiv 0$, $H(u_q)\equiv 0$, $I(u_p,u_q)=0$ are conditions that u_p and u_q be powers of the same linear form, $I(u_p,u_q)$ being any simultaneous invariant of u_p and u_q.

THEOREM. *Necessary and sufficient conditions that a curve of the n-th degree have an n-fold point are (when the equation of the curve has been brought into the form* (16))

$$H(u_2)\equiv H(u_3)\equiv\ldots.\equiv H(u_n)\equiv 0,$$

together with a set obtained by equating to zero any simultaneous invariants of a non-vanishing u with each of the other u's.

In particular, then, the conditions for a triple point on the cubic

$$x_3^3 - 3 f x_3 + 2 \phi = 0$$

will be

$$D = 0, \quad \Delta \equiv 0, \quad F = 0.$$

The ternary condition for an n-fold point is the identical vanishing of the Hessian. This gives rise to $\frac{1}{2}(3n-5)(3n-4)$ equations among the coefficients. The set of conditions above give rise to but $(n-1)^2 + n - 2 = n^2 - n - 1$.*

The number of conditions for the conic, cubic, quartic and quintic in the two cases are: 1, 10, 28, 55, and 1, 5, 11, 19 respectively.

MIDDLETOWN, CONN., *April*, 1912.

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* It is not intended here to imply that this is a minimum set of conditions. Cf. O. E. Glenn, "On Semi-discriminants of Ternary Forms," *Transactions of the American Mathematical Society,* Vol. XII, No. 3 (1911), pp. 367–374.